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C^*-simplicity and Burnside quotients

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Young Geometric Group Theory XIII

Characterizations of C^*-simplicity

The primary objective of this project is to construct new C^*-simple groups, which play a fundamental role in operator algebras. A discrete group G is said to be C^***-simple** if its reduced C^*-algebra $C_r^*(G)$ is simple, meaning it has no non-trivial ideals.

Recently, an intrinsic group-theoretic characterization of C^*-simplicity was established, based on the notion of residually normal subgroups. Given a discrete group G, a subgroup $H < G$ is said to be **residually normal** if there exists a finite subset $F \subseteq G \setminus \{e\}$ such that $F \cap gHg^{-1} \neq \varnothing$ for all $g \in G$.

> A discrete group is C^*-simple **if and only if** it has no amenable residually normal subgroups [3].

We plan to investigate C^*-simplicity of free Burnside groups and related constructions.

Free Burnside groups

The **free Burnside group** of rank $m \geq 2$ and exponent n is defined as

$$B(m,n) = F_m / \langle\langle w^n \mid w \in F_m \rangle\rangle,$$

where F_m is the free group on m generators. A fundamental question in group theory, known as the **Burnside problem**, asks whether the free Burnside group $B(m,n)$ is finite. The first solution was given in 1968 by Adyan and Novikov [4], who proved that $B(m,n)$ is infinite for $m \geq 2$ and odd $n \geq 4381$. Subsequent works lowered the bound for odd exponents and addressed the case of even exponents.

In 2023, Agatha Atkarskaya, Eliyahu Rips and Katrin Tent were able to show that $B(m,n)$ is infinite for $m \geq 2$ and odd $n \geq 557$, which is the best currently known lower bound for the exponent [2].

The proof is based on techniques of iterated **small cancellation theory**, where the induction is based on the nesting depth of relators. A key step is defining a **canonical form** for cosets in $B(m,n)$. Given a word $w \in F_m$, one inductively constructs a sequence

$$w \mapsto \mathsf{can}_1(w) \mapsto \mathsf{can}_2(\mathsf{can}_1(w)) \mapsto \dots,$$

which stabilizes after finitely many steps. The resulting word, denoted by $\mathsf{can}(w)$, is the canonical form of w. Then, two words w and v represent the same element in $B(m,n)$ if and only if $\mathsf{can}(w) = \mathsf{can}(v)$.

First steps

The first phase of this project focuses on using the canonical form to provide accessible proofs of key structural results on $B(m,n)$, previously known for large n through highly intricate arguments. Specifically, we show that, for $m \geq 2$ and odd $n \geq 557$, the following hold.

> - Every non-abelian subgroup of $B(m,n)$ contains a copy of $B(2,n)$.
> - Every abelian subgroup of $B(m,n)$ is cyclic.

Thanks to the intrinsic characterization of C^*-simplicity, understanding the subgroup structure of $B(m,n)$ is key to proving that these groups are C^*-simple - a result already established through different methods for sufficiently large odd n. This strategy relies on the fact that $B(2,n)$ is non-amenable, as proven by Adyan. Furthermore, our proof for C^*-simplicity naturally adapts to the more general setting of free products of C^*-simple groups.

Burnside quotients of free products and HNN extensions

We aim to extend our approach to Burnside-like quotients of fundamental group constructions, such as free products and HNN extensions, where the arguments for C^*-simplicity will need to be adjusted accordingly. Given a group G, the term **Burnside quotient** typically refers to the group obtained as

$$G / \langle\langle w^n \mid w \in G \rangle\rangle,$$

where additional conditions on the relators may be imposed to define weaker variants.

- **Free products:** we consider groups of the form

  $$G * H / \langle\langle w^n \mid w \notin \bigcup_{g \in G*H} G^g \cup H^g \rangle\rangle.$$

  Notably, setting $G = H = \mathbb{Z}/n\mathbb{Z}$ here recovers the free Burnside group $B(2,n)$.
- **HNN extensions:** investigating these constructions is crucial in building non-split sharply 2-transitive groups of smaller positive characteristic, as the procedure involves taking Burnside-like quotients of HNN extensions [1].

References

[1] M. Amelio, S. André, and K. Tent. Non-split sharply 2-transitive groups of odd positive characteristic, 2023.

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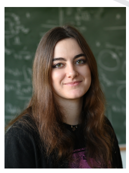
[3] M. Kennedy. An intrinsic characterization of C^*-simplicity. *Annales scientifiques de l'École normale supérieure*, 2015.

[4] P. S. Novikov and S. I. Adyan. On infinite periodic groups. I–III. *Math. USSR, Izv.*, 2:209–236, 241–480, 665–685, 1969.



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